

## PROGRESS IN NONEQUILIBRIUM QUANTUM FIELD THEORY II \*

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We review recent developments for the description of nonequilibrium quantum fields, continuing hep-ph/0302210.<sup>1</sup>

### 1. Motivations

Nonequilibrium quantum fields play a central role in various areas of physics, such as cosmology, ultra-relativistic heavy-ion collisions, or condensed matter physics. A timely example in cosmology concerns the issue of reheating<sup>2</sup> of the early universe: At the end of inflation the universe is left in a frozen albeit very high-energy state, very different from the subsequent hot universe observed from the cosmic microwave background. This amounts to an enormous entropy growth due to particle production. It is important progress that questions such as the far-from-equilibrium particle production at the end of the inflationary universe, including the subsequent process of thermalization, can be addressed in quantum field theory from first principles.<sup>3</sup> The late-time approach to quantum thermal equilibrium represents a complex collective phenomenon, which can occur on time scales dramatically larger than typical relaxation times.<sup>4,5,6</sup> For phenomenological applications it is therefore often crucial that different quantities effectively thermalize on different time scales and a complete thermalization of all quantities may not be necessary. For instance, an approximately time-independent equation of state  $p = p(\epsilon)$ , characterized by an almost fixed relation between pressure  $p$  and energy density  $\epsilon$ , can form very early – even though the system is still far from equilibrium. Such prethermalized quan-

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ties approximately take on their final thermal values already at a time when the occupation numbers of individual momentum modes still show strong deviations from the late-time Bose-Einstein or Fermi-Dirac distribution. Prethermalization is a universal far-from-equilibrium phenomenon which occurs on time scales dramatically shorter than the thermal equilibration time.<sup>7</sup> In order to establish such a behavior it is crucial to be able to compare between the time scales of prethermalization and thermal equilibration. Approaches based on small deviations from equilibrium, or on a sufficient homogeneity in time underlying kinetic descriptions, are not applicable in this case to describe the required “link” between the early and the late-time behavior.

Similar issues arise in the study of high-energy nuclear collisions, where highly excited matter is produced far from equilibrium. The experiments seem to indicate early thermalization whereas the present theoretical understanding of QCD suggests a much longer thermal equilibration time,<sup>8</sup> thereby questioning for instance the picture of a weakly interacting partonic gas.<sup>9</sup> This also points to the need to go beyond kinetic approaches, which do not provide a valid description on very short time scales. Efforts to go beyond kinetic descriptions include studies of the non-equilibrium dynamics of a chiral quark-meson model,<sup>5,7</sup> or photon production<sup>10</sup> in heavy ion collisions.

The recent progress in the theoretical description of nonequilibrium quantum fields is based on efficient functional integral techniques, so-called  $n$ -particle irreducible effective actions, for which powerful nonperturbative approximation schemes are available.<sup>11</sup> The techniques provide an efficient approximation scheme also for situations close to, or in equilibrium where resummations are needed. This includes e.g. the calculation of transport coefficients<sup>12</sup> and thermodynamic quantities<sup>13</sup> in hot field theories or the study of critical phenomena near second-order phase transitions.<sup>14</sup> These are promising systematic approaches to cure the convergence problems of perturbative calculations of thermodynamic quantities at high temperature such as pressure or entropy.<sup>15</sup>

Though we focus on particle physics and cosmology applications, we emphasize that these techniques can be equally applied to other nonequilibrium phenomena in complex many body systems. A prominent example is the study of Bose-Einstein condensates. The interpretation of various recent experiments, for instance concerning Bose condensation of molecules,<sup>16</sup> indicate important contributions beyond standard mean-field type approximations for the real-time dynamics of correlations.<sup>17</sup> These can be system-

atically taken into account using  $n$ PI effective action techniques.<sup>18</sup>

Apart from these applications, there has been much progress in understanding formal aspects of approximation schemes based on  $n$ PI effective actions, such as renormalization and the validity of Ward identities for gauge theories. In the following we review these developments.

## 2. Non-equilibrium quantum field theory: methods

### 2.1. *Standard approximations fail out of equilibrium*

Nonequilibrium dynamics requires the specification of an initial state at some time  $t = 0$ , characterized by an initial density matrix  $\rho_D$ . All the information about the time-evolution of the system is encoded in the correlation functions of field operators  $\hat{\varphi}$ , which may be obtained from the generalized generating functional:

$$Z[R_1, R_2, \dots; \rho_D] = \text{Tr} \left\{ \rho_D e^{i \sum_n R_n \cdot \hat{\varphi}^n} \right\}, \quad (1)$$

with  $n$ -point classical sources  $R_n \equiv R_n(x_1, \dots, x_n)$ . Here,  $R_n \cdot \hat{\varphi}^n \equiv \int_{x_1 \dots x_n} R_n(x_1, \dots, x_n) \hat{\varphi}(x_1) \cdots \hat{\varphi}(x_n)$ , with  $\int_x \equiv \int_{\mathcal{C}} dx^0 \int d^3x$ . The closed time contour  $\mathcal{C}$ ,<sup>19</sup> runs from the initial time  $t = 0$ , where the system has been prepared, to an arbitrarily large time, and backward to the initial time. The functional (1) can be given the following path-integral representation:

$$Z[R_1, R_2, \dots; \rho_D] = \int d\varphi_0 d\varphi'_0 \langle \varphi_0 | \rho_D | \varphi'_0 \rangle \int_{\varphi_0}^{\varphi'_0} \mathcal{D}\varphi e^{i(\mathcal{S}[\varphi] + \sum_n R_n \cdot \varphi^n)} \quad (2)$$

where  $\mathcal{S}[\varphi]$  is the classical action and  $\int_{\varphi_0}^{\varphi'_0} \mathcal{D}\varphi$  represents the integral over classical paths along the contour  $\mathcal{C}$ , with fixed boundaries on both ends at  $t = 0$  corresponding to the field configurations  $\varphi_0(\mathbf{x})$  and  $\varphi'_0(\mathbf{x})$ . One finally performs the appropriate average over possible initial conditions. All correlation functions can be given similar expressions by functional differentiation. A direct evaluation of the corresponding path integrals in real time is currently not possible with standard Monte-Carlo techniques. Equivalently, the solution of the real-time dynamics of the functional Schrödinger equation is prohibitively difficult. A frequently employed strategy is to concentrate on classical statistical field theory instead, which can be simulated. This can give important insights when the number of field quanta per mode is sufficiently large such that quantum fluctuations are suppressed compared to statistical fluctuations. However, classical Rayleigh-Jeans divergences and the lack of genuine quantum effects – such as the approach

to quantum thermal equilibrium characterized by Bose-Einstein or Fermi-Dirac statistics – limit their use. A coherent understanding of the time evolution in quantum field theory is required – a program which has made substantial progress in recent years with the development of powerful theoretical techniques.

There exist various powerful approximation methods for vacuum field theory as well as for thermal equilibrium at nonzero temperature. However, most standard field theoretical techniques are not applicable for out-of-equilibrium situations, where one encounters additional complications. The first new aspect concerns secularity: The perturbative time evolution suffers from the presence of spurious, so-called secular terms, which grow with time and invalidate the expansion even in the presence of a weak coupling. Here it is important to note that the very same problem appears as well for nonperturbative approximation schemes such as standard  $1/N$  expansions, where  $N$  denotes the number of field components.<sup>a</sup> Typically, secularity is not a very difficult problem and for a given approximation there can be various ways to resolve it. There is a requirement, however, which poses very strong restrictions on the possible approximations: Universality, i.e. the insensitivity of the late-time behavior to the details of the initial conditions. If thermal equilibrium is approached then the late-time result is universal in the sense that it becomes uniquely determined by the energy density and further conserved charges. To implement the necessary nonlinear dynamics is demanding. Both requirements of a non-secular and universal behavior can indeed be fulfilled using efficient functional integral techniques: so-called  $n$ -particle irreducible effective actions, where the simplest example is given by the two-particle-irreducible (2PI) effective action,<sup>20</sup> which is described in the following.

### 2.1.1. *Secularity*

Here, we discuss the well-known secularity problem in the context of a relativistic real scalar field theory with potential  $m^2\varphi^2/2 + \lambda\varphi^4/4!$ . For the argument it is irrelevant that we assume spatial homogeneity and isotropy as well as  $Z_2$ -symmetry, so that  $\text{Tr}\{\rho_D \hat{\varphi}(x)\} = 0$ , with Gaussian initial density matrix  $\rho_D$ .

<sup>a</sup>Note that restrictions to mean-field type approximations such as leading-order large- $N$  are insufficient. They suffer from the presence of an infinite number of spurious conserved quantities and are known to fail to describe thermalization. Secularity enters the required next-to-leading order corrections and beyond.

The time-ordered propagator along the contour  $\mathcal{C}$  is defined as:

$$G(x, y) = \text{Tr}\{\rho_D T_{\mathcal{C}}(\hat{\varphi}(x) \hat{\varphi}(y))\} = \Theta_{\mathcal{C}}^+ G^>(x, y) + \Theta_{\mathcal{C}}^- G^<(x, y), \quad (3)$$

where  $\Theta_{\mathcal{C}}^{\pm} \equiv \Theta_{\mathcal{C}}(\pm(x^0 - y^0))$ . One obtains to  $\mathcal{O}(\lambda^2)$ :

$$G_{2\text{-loop}}(x, y) = G_0(x - y) + \int_{uv} G_0(x - u) \Sigma_2(u - v) G_0(v - y), \quad (4)$$

where the bare inverse propagator is given by  $G_0^{-1}(x, y) = i(\square + m^2)\delta_{\mathcal{C}}(x - y)$  and where  $\Sigma_2$  denotes the self-energy to two-loop order:  $\Sigma_2(x - y) = -i\lambda G_0(x, x)\delta_{\mathcal{C}}(x - y)/2 - \lambda^2 G_0(x - y)/6$ . The first (one-loop) contribution corresponds to a simple mass shift and plays no role for the present argument. We ignore it as well as all local insertions in the following discussion. We therefore write:

$$\Sigma_2(x - y) = -\frac{\lambda^2}{6} G_0^3(x - y). \quad (5)$$

For illustrational purposes, we introduce an effective particle number  $n_p(t)$ , defined as:

$$\omega_p (2n_p(t) + 1) = [\partial_t \partial_{t'} G^<(t, t'; p) + \omega_p^2 G^<(t, t'; p)]_{t'=t}, \quad (6)$$

where  $\omega_p = \sqrt{p^2 + m^2}$  and where  $G(t, t'; p)$  denotes the spatial Fourier transform of the propagator. Writing explicitly the time integrals in (4), one easily obtains, for the two-loop contribution to the particle number:

$$n_p^{2\text{-loop}}(t) - n_p^0 = (1 + n_p^{(0)})\mathcal{R}_p^<(t) - n_p^{(0)}\mathcal{R}_p^>(t), \quad (7)$$

where, introducing the Fourier transform in frequency space  $\bar{\Sigma}_2^{<, >}(\omega, p)$ :

$$\mathcal{R}_p^{<, >}(t) = -\frac{1}{\omega_p} \int \frac{d\omega}{2\pi} \bar{\Sigma}_2^{<, >}(\omega, p) \frac{1 - \cos(\omega - \omega_p)t}{(\omega - \omega_p)^2}. \quad (8)$$

In the large-time limit, one gets:

$$\mathcal{R}_p^{<, >}(t) \simeq \bar{\sigma}_p^{<, >} \times t, \quad (9)$$

with the on-shell rates:  $\bar{\sigma}_p^{<, >} = -\bar{\Sigma}_2^{<, >}(\omega_p, p)/2\omega_p$ . We recover the usual secular behavior of perturbation theory, which occurs whenever the rates in (9) do not vanish on-shell. For instance, in the present two-loop approximation, the production rate  $\bar{\sigma}_p^<$  receives contributions from in-medium two-body scatterings  $\mathbf{q} + \mathbf{k} \rightarrow \mathbf{r} + \mathbf{p}$ :

$$\bar{\sigma}_p^< = \frac{\lambda^2}{4\omega_p} \int \tilde{d}q \tilde{d}k \tilde{d}r (2\pi)^4 \delta^{(4)}(Q + K - R - P) \times n_q^0 n_k^0 (1 + n_r^0). \quad (10)$$

where we introduced four-momenta as *e.g.*  $Q^\mu \equiv (\omega_q, \mathbf{q})$  and  $\tilde{d}q \equiv \frac{d^3q}{(2\pi)^3 2\omega_q}$ .

A similar analysis reveals that higher-order contributions to Eq. (4) are secular as well. The standard solution to this problem corresponds to summing the infinite series of self-energy insertions:

$$\begin{aligned} G_2 &= G_0 + G_0 \cdot \Sigma_2 \cdot G_0 + G_0 \cdot \Sigma_2 \cdot G_0 \cdot \Sigma_2 \cdot G_0 + \dots \\ &= G_0 + G_0 \cdot \Sigma_2 \cdot G_2, \end{aligned} \quad (11)$$

where we have used an obvious schematic notation. This corresponds to the usual resummation in terms of one-particle-irreducible (1PI) proper vertices. In the two-loop example discussed here, this leads to a non-secular time-evolution. However, the problem reappears for higher order contributions to the self-energy. For instance, the following  $\mathcal{O}(\lambda^4)$  contribution

$$\frac{\lambda^4}{36} G_0^2(x-y) \times \int_{uv} G_0(x-u) G_0^3(u-v) G_0(v-y) \quad (12)$$

contains the same time-integral as analyzed above and thus diverges at large times. Therefore, the systematic resolution of the secularity problem requires infinite resummations beyond the standard 1PI resummation scheme.

To further elaborate the discussion, we consider the sum of contributions (5) and (12), which can be written as:  $\Sigma_4 = -(\lambda^2/6) G_0^2 \{G_0 + G_0 \cdot \Sigma_2 \cdot G_0\}$ . Our previous analysis suggests to resum the infinite geometric series of self-energy ( $\Sigma_2$ ) insertions in the term in brackets:  $-(\lambda^2/6) G_0^2 \{G_0 + G_0 \cdot \Sigma_2 \cdot G_0 + \dots\} = -(\lambda^2/6) G_0^2 G_2$ . Similarly, one can resum self-energy insertions on the other two lines of the original two-loop diagram, to arrive at the following simple – and suggestive – expression:

$$\Sigma_{[2]} = -\frac{\lambda^2}{6} G_2^3. \quad (13)$$

To improve on this, we note that the simple expression (13), suggests the following iterative procedure to include higher orders without encountering new secular terms: From the propagator at the  $n$ -th iteration  $G_{[2n]}$ , compute the resummed two-loop self-energy  $\Sigma_{[2n]} = -\lambda^2 G_{[2n]}^2/6$ . The next iteration  $G_{[2(n+1)]}$  is then computed from  $G_{[2(n+1)]}^{-1} = G_0^{-1} - \Sigma_{[2n]}$ . Starting from  $G_{[0]} = G_0$ , this converges, after infinitely many iterations, to the

following equation for  $G \equiv G_{[\infty]}$ :<sup>b</sup>

$$G^{-1} = G_0^{-1} - \Sigma[G] \quad (14)$$

with

$$\Sigma[G] = -\frac{\lambda^2}{6} G^3. \quad (15)$$

We arrive at a non-linear equation for the two-point function  $G$ . The procedure described above guarantees that possible secular terms are automatically resummed at each step of the iteration. Equation (15) is remarkably simple and suggests to reorganize the original perturbative expansion in terms of the full propagator. Before we discuss below that this is efficiently described in terms of the 2PI effective action, we describe in the following that this procedure also automatically respects late-time universality.

### 2.1.2. Universality

The price to be paid for the above non-secular scheme is that at some order one necessarily has to solve nonlinear equations without further approximations. However, as we will see next, it is precisely the nonlinearity which is required to be able to obtain universality in the sense mentioned above. We consider first Eq. (11), which is more conveniently rewritten for initial-value problems as:  $G_0^{-1} \cdot G_2 = 1 + \Sigma_2 \cdot G_2$ . Using this equation, evaluated along the contour  $\mathcal{C}$ , one obtains for the time-derivative of the effective particle number  $n_p(t)$ :

$$\begin{aligned} \omega_p \dot{n}_p(t) = \text{Re} \int_0^t dt' \left\{ \Sigma_2^<(t, t'; p) (i\partial_t - \omega_p) G_2^>(t', t; p) \right. \\ \left. - \Sigma_2^>(t, t'; p) (i\partial_t - \omega_p) G_2^<(t', t; p) \right\}. \end{aligned} \quad (16)$$

To illustrate the content of this equation, we employ the following free-field like ansatz:  $(i\partial_t - \omega_p) G_2^<(t', t; p) \rightarrow -n_p(t) e^{-i\omega_p(t'-t)}$ , and similarly for  $G_2^>$  with  $n_p(t)$  replaced by  $1 + n_p(t)$ . Using (8)) one finds:

$$\dot{n}_p(t) = (1 + n_p(t)) \dot{\mathcal{R}}_p^<(t) - n_p(t) \dot{\mathcal{R}}_p^>(t). \quad (17)$$

<sup>b</sup>This procedure is a direct generalization of the successive summations of so-called daisy and super-daisy diagrams, which correspond to the standard Hartree approximation. With present notations, the latter corresponds to iterating the one-loop self-energy:  $\Sigma_{[n]}(x, x') = -i\lambda G_{[n]}(x, x) \delta_{\mathcal{C}}(x - x')/2$ . In the discussion presented here, we have discarded these (trivial) one-loop tadpole insertions for simplicity.

Using the late-time behavior (9), one finally gets:

$$n_p(t) \simeq n_p^0 e^{-\gamma_p t} + \frac{\bar{\sigma}_p^<}{\gamma_p} \left(1 - e^{-\gamma_p t}\right) \rightarrow \frac{\bar{\sigma}_p^<}{\gamma_p}, \quad (18)$$

where  $\gamma_p = \bar{\sigma}_p^> - \bar{\sigma}_p^<$ . The late-time limit, indicated on the RHS, is completely determined by the initial particle number  $n_p^0$ , see (10). This dependence on initial conditions is rooted in the fact that Eq. (11) is a linear equation for the propagator. Late-time universality – and the associated effective loss of information – requires nonlinear equations.

To illustrate the importance of the resummation procedure discussed in Sec. 2.1.1 and, in particular, of the non-linear character of the resulting equations, we consider the time evolution of the particle number (6) for the approximation (15). We emphasize that these particle numbers are only employed for illustrational purposes. The quantum field theoretical description for the propagator contains off-shell as well as memory effects and is not limited to late-times.<sup>4</sup> Moreover, the functional formulation of this approach, based on the 2PI effective action described below provides a systematic framework for higher-order calculations. Using Eq. (14), it is easy to check that, in the present approximation, the time derivative of  $n_p(t)$  has a similar form as in (16) with the replacements  $G_2^{<,>} \rightarrow G^{<,>}$  and  $\Sigma_2^{<,>} \rightarrow \Sigma^{<,>}$ . Employing a similar free-field like ansatz as above, one obtains the following Boltzmann-like equation at sufficiently large times:

$$\begin{aligned} \dot{n}_p(t) = & \frac{\lambda^2}{4\omega_p} \int \tilde{d}q \tilde{d}k \tilde{d}r (2\pi)^4 \delta^{(4)}(Q + K - R - P) \\ & \times \left\{ n_q(t) n_k(t) (1 + n_r(t)) (1 + n_p(t)) \right. \\ & \left. - n_p(t) n_r(t) (1 + n_k(t)) (1 + n_q(t)) \right\}. \end{aligned} \quad (19)$$

The late-time equilibrium solution - corresponding to  $\dot{n}_p(t) = 0$  - is given by the Bose-Einstein distribution:

$$n_p^{\text{eq}} = \frac{1}{e^{(\omega_p - \mu)/T} - 1}, \quad (20)$$

and is only characterized by the temperature  $T$  and chemical potential  $\mu$ . The latter are uniquely determined by the total energy and particle number densities, which are conserved in the present approximation.<sup>c</sup> As expected,

<sup>c</sup>While the former is an exactly conserved quantity, the latter is only conserved in the approximation (19), which describes on-shell two-body elastic scatterings. In particular, it is not conserved by the original equation (14), which includes off-shell, particle-number-



the late-time result is insensitive to the details of the initial condition. Note that, in contrast, resummation schemes based on local insertions, such as *e.g.* the so-called 2PPI loop expansion, fail to describe late-time thermalization at finite loop order.<sup>21</sup> This demonstrates that it is crucial to go beyond local (mass) resummations.

## 2.2. The 2PI effective action

The previous analysis illustrates that a systematic way of respecting both non-secularity as well as universality is to reorganize the perturbative expansion in terms of the full propagator. This can be done in an efficient way by performing a double Legendre transform of the functional (1) with respect to the linear and bilinear sources  $R_1(x)$  and  $R_2(x, y)$ . One then obtains the so-called two-particle-irreducible (2PI) effective action, a functional of the one and two-point functions  $\phi(x)$  and  $G(x, y)$ , which can be conveniently parametrized as:<sup>20</sup>

$$\Gamma[\phi, G] = S[\phi] + \frac{i}{2} \text{Tr} \ln G^{-1} + \frac{i}{2} \text{Tr} G_0^{-1} \cdot G + \Gamma_2[\phi, G], \quad (21)$$

where  $\Gamma_2$  can be written as an infinite series of closed 2PI diagrams with lines corresponding to  $G$  and vertices given by the shifted action  $\mathcal{S}[\phi + \varphi]$ . Physical solutions corresponds to the stationarity conditions:

$$\frac{\delta \Gamma[\phi, G]}{\delta \phi} = 0 \quad , \quad \frac{\delta \Gamma[\phi, G]}{\delta G} = 0. \quad (22)$$

Using (21), the second of these equations is equivalent to the equation of motion (14) for  $G$ , with self-energy given by:

$$\Sigma[\phi, G] = 2i \frac{\delta \Gamma_2[\phi, G]}{\delta G}. \quad (23)$$

Thus, one automatically obtains non-linear, self-consistent equations, as required to describe late-time thermalization. This generalizes the resummation procedure described previously.<sup>d</sup> Systematic approximations include a loop or a  $1/N$ -expansion of the 2PI effective action beyond leading order.<sup>22</sup>

Finally, the effective action  $\Gamma[\phi]$ , which encodes all the  $n$ -point functions of the theory, is obtained as:

$$\Gamma[\phi] = \Gamma[\phi, G[\phi]], \quad (24)$$

changing processes. When the latter are taken into account, the equilibrium distribution is given by (20) with a vanishing chemical potential.<sup>4</sup>

<sup>d</sup>For instance, the resummed two-loop self-energy (15) is obtained from the three-loop contribution to the functional  $\Gamma_2$ :  $i(\lambda^2/48) \int_{xy} G^4(x, y)$ .

where  $G[\phi]$  is the solution of the second of Eqs. (22). Equation (24) is an exact relation between the 1PI and 2PI functionals which merely states that both are equivalent representations of the theory in the absence of higher-than-linear sources  $R_{n \geq 2}$ . However, systematic (*e.g.* loop or  $1/N$ ) expansions of the 1PI and 2PI functionals do not coincide order by order in general. Equation (24) therefore defines a non-trivial approximation scheme for the computation of  $n$ -point functions. We call it the 2PI resummation scheme. It is non-perturbative in nature and, as we have argued previously, it includes all the ingredients needed to overcome both the secularity and universality problems of non-equilibrium quantum field theory.

### 2.3. Symmetries

Linear symmetries of the classical action are directly promoted at the level of the 2PI effective action. For example, for a  $O(N)$ -symmetric scalar field theory,  $\Gamma[\phi, G]$  is invariant under the transformation:

$$\phi(x) \rightarrow \mathcal{R}\phi(x) \quad ; \quad G(x, y) \rightarrow \mathcal{R}G(x, y)\mathcal{R}^\dagger, \quad (25)$$

where  $\mathcal{R}$  denotes a  $O(N)$  rotation. This ensures that the resummed effective action (24) has the same symmetry properties as the classical action:  $\Gamma[\phi] = \Gamma[\mathcal{R}\phi]$ . Notice that this holds for any 2PI approximation which respect the symmetry. In turn, this directly implies that Ward identities are automatically satisfied by the  $n$ -point functions calculated from the resummed effective action (24). For instance, in the case of spontaneously broken symmetry, the two-point function, evaluated at the solution  $\phi = \bar{\phi}$  of the first of Eqs. (22):

$$\Gamma_{ab}^{(2)}(x, y) = \left. \frac{\delta^2 \Gamma[\phi]}{\delta \phi_a(x) \delta \phi_b(y)} \right|_{\phi = \bar{\phi}}, \quad (26)$$

where  $a, b$  are  $O(N)$  indices, exactly satisfies Goldstone's theorem<sup>e</sup>.<sup>22,24</sup>

We emphasize that similar arguments can be applied to arbitrary linear and/or affine symmetry transformations including local ones. For instance, for QED in linear gauges, one can show that the 2PI resummed effective action satisfies the usual Ward identities. In particular, one immediately

<sup>e</sup>In contrast, the solution  $G_{ab}[\bar{\phi}]$  of the second of Eqs. (22) violates Goldstone's theorem in general. However, it can be shown that for systematic expansion schemes, such as a 2PI loop or  $1/N$ -expansion at a given order, the functions (26) and  $iG_{ab}^{-1}[\bar{\phi}]$  only differ by higher-order terms and, therefore, these violations are of higher order as well.<sup>23</sup>

concludes that the photon polarization tensor calculated from it is transverse in momentum space at any order of a 2PI loop-expansion.<sup>23</sup> The generalization to nonabelian gauge theories is technically more involved and needs to be further investigated.

#### 2.4. Renormalization

A systematic renormalization procedure of the resummed propagator at vanishing field,  $G[\phi = 0]$ , has been proposed in Refs. <sup>24,25</sup> for the 2PI loop-expansion in scalar field theories.<sup>f</sup> This is based on applying a BPHZ subtraction procedure to diagrams with resummed propagators. In particular, it has been shown that the subtleties arising from self-consistent resummations reduce in this case to a proper renormalization of the following Bethe-Salpeter like equation (here,  $G \equiv G[\phi = 0]$ ):

$$\bar{V}(x, y, z, t) = \bar{\Lambda}(x, y, z, t) + \frac{i}{2} \int_{uv\bar{u}\bar{v}} \bar{\Lambda}(x, y, u, v) G(u, \bar{u}) G(v, \bar{v}) \bar{V}(\bar{u}, \bar{v}, z, t), \quad (27)$$

where

$$\bar{\Lambda}(x, y, z, t) \equiv 4 \frac{\delta^2 \Gamma_2[\phi = 0, G]}{\delta G(x, y) \delta G(z, t)} \Big|_{G=G[\phi=0]}. \quad (28)$$

This analysis has been recently generalized in <sup>13,23</sup> to include the field-dependence of the 2PI resummed effective action (24). This allows one to determine all the counterterms of the theory at a given level of approximation from suitable renormalization conditions and to obtain finite results for arbitrary  $n$ -point functions. As an explicit example, we consider the one-component scalar field theory discussed in the previous sections at order  $\lambda_R^2$  in the 2PI coupling-expansion, where  $\lambda_R$  denotes the renormalized coupling. We employ renormalization conditions for the two-point function,  $\Gamma^{(2)}$ , and four-point function,  $\Gamma^{(4)}$ , given by:

$$\Gamma^{(2)}(x, y) \equiv \frac{\delta^2 \Gamma[\phi]}{\delta \phi(x) \delta \phi(y)} \Big|_{\phi=0}, \quad (29)$$

$$\Gamma^{(4)}(x, y, z, t) \equiv \frac{\delta^4 \Gamma[\phi]}{\delta \phi(x) \delta \phi(y) \delta \phi(z) \delta \phi(t)} \Big|_{\phi=0}. \quad (30)$$

Without loss of generality we use renormalization conditions for  $\phi = 0$

<sup>f</sup>For a recent application to the 2PI  $1/N$ -expansion, see <sup>26</sup>.

which in Fourier space read:

$$Z \Gamma^{(2)}(p^2)|_{p=0} = -m_R^2, \quad (31)$$

$$Z \frac{d}{dp^2} \Gamma^{(2)}(p^2)|_{p=0} = -1, \quad (32)$$

$$Z^2 \Gamma^{(4)}(p_1, p_2, p_3)|_{p_1=p_2=p_3=0} = -\lambda_R, \quad (33)$$

with the wave function renormalization  $Z$ . Defining the renormalized fields as:  $\phi_R = Z^{-1/2}\phi$  and  $G_R[\phi_R] = Z^{-1}G[\phi]$ , and introducing the usual counterterms:  $Zm^2 = m_R^2 + \delta m^2$ ,  $Z^2\lambda = \lambda_R + \delta\lambda$  and  $\delta Z = Z - 1$ , the classical contribution to the effective action (24) reads:

$$S[\phi] = \int_x \left( \frac{1 + \delta Z}{2} \partial_\mu \phi_R \partial^\mu \phi_R - \frac{m_R^2 + \delta m^2}{2} \phi_R^2 - \frac{\lambda_R + \delta\lambda}{24} \phi_R^4 \right). \quad (34)$$

Similarly, one can write for the one-loop part:  $\text{Tr} \ln G^{-1}[\phi] = \text{Tr} \ln G_R^{-1}[\phi_R]$  up to an irrelevant constant, and:

$$\begin{aligned} \frac{i}{2} \text{Tr} G_0^{-1}[\phi] G[\phi] &= -\frac{1}{2} \int_x [(1 + \delta Z_1) \square_x + m_R^2 + \delta m_1^2] G_R(x, y; \phi_R)|_{x=y} \\ &\quad - \frac{\lambda_R + \delta\lambda_1}{4} \int_x \phi_R^2(x) G_R(x, x; \phi_R). \end{aligned} \quad (35)$$

Here,  $\delta Z_1$ ,  $\delta m_1^2$  and  $\delta\lambda_1$  denote the same counterterms as above, however approximated to the given order. Finally, the contribution from the 2PI functional  $\Gamma_2$  in Eq. (21) reads, at order  $\lambda_R^2$ :

$$\begin{aligned} \Gamma_2[\phi, G[\phi]] &= -\frac{\lambda_R + \delta\lambda_2}{8} \int_x G_R^2(x, x; \phi_R) + i \frac{\lambda_R^2}{48} \int_{xy} G_R^4(x, y; \phi_R) \\ &\quad + i \frac{\lambda_R^2}{12} \int_{xy} \phi_R(x) G_R^3(x, y; \phi_R) \phi_R(y). \end{aligned} \quad (36)$$

One first has to calculate the solution  $G_R[\phi_R]$  of the stationarity condition for the 2PI effective action. For this, one has to impose the same renormalization conditions as for the propagator (31)-(32) in Fourier space:

$$i G_R^{-1}(p^2; \phi_R)|_{p=0, \phi_R=0} = -m_R^2, \quad (37)$$

$$\frac{d}{dp^2} i G_R^{-1}(p^2; \phi_R)|_{p=0, \phi_R=0} = -1, \quad (38)$$

together with the condition:

$$\bar{V}_R(p_1, p_2, p_3)|_{p_1=p_2=p_3=0} = -\lambda_R, \quad (39)$$

for the renormalized “four-point” field, obtained from Eq. (27). It is easy to check that conditions (37)-(39) fix all the counterterms of the effective

action at vanishing field, namely  $\delta Z_1$ ,  $\delta m_1^2$  and  $\delta \lambda_2$ . This corresponds to the case considered in <sup>24,25</sup>.

The remaining counterterms are to be determined from the renormalization conditions (31)-(33). A similar analysis to the one performed in <sup>25</sup> shows that the coupling divergences appearing in the two-point function (29) can be absorbed in a proper renormalization of the following four-point field:<sup>23</sup>

$$V_R(x, y, z, t) \equiv \frac{\delta^2 iG_R^{-1}(x, y; \phi_R)}{\delta \phi_R(z) \delta \phi_R(t)} \Big|_{\phi_R=0}, \quad (40)$$

which satisfies a Bethe-Salpeter like equation similar to (27). The corresponding condition reads, in momentum space:

$$V_R(p_1, p_2, p_3) \Big|_{p_1=p_2=p_3=0} = -\lambda_R. \quad (41)$$

The latter, together with Eqs. (31)-(32) fix the counterterms associated with the quadratic field-dependence in Eqs. (34)-(36), namely  $\delta Z$ ,  $\delta m^2$  and  $\delta \lambda_1$ .<sup>8</sup> Finally, the remaining counterterm  $\delta \lambda$ , associated with the quartic field-dependence, is determined from Eq. (33).

It is important to emphasize that Eqs. (37) and (38), as well as (39) and (41) are not independent conditions. Instead, they are fixed by the renormalization conditions (31)-(33). This express the fact that in the exact theory, the following relations between renormalized quantities hold:  $\Gamma_R^{(2)} = iG_R^{-1}(\phi_R = 0)$  and  $\Gamma_R^{(4)} = \bar{V}_R = V_R$ . Thus, the renormalization procedure described here provides an efficient fixing of all the above counterterms, including those associated with the explicit field-dependence of the effective action. The latter play a crucial role *e.g.* in the broken phase and are essential for the determination of the effective potential.

## 2.5. Equivalence hierarchy for $n$ PI effective actions

The 2PI resummation scheme provides a self-consistent dressing of the field  $\phi$  and the propagator  $G$ . This can be generalized to higher  $n$ -point functions as well: The  $n$ PI effective action  $\Gamma[\phi, G, V_3, \dots, V_n]$  provides a self-consistent description for the dressed proper vertices  $V_3, \dots, V_n$ , which are obtained the stationarity conditions:  $\delta \Gamma / \delta V_3 = 0, \dots, \delta \Gamma / \delta V_n = 0$ . As

<sup>8</sup>In the present approximation, one has  $\delta Z_1 = \delta Z$  and  $\delta m_1^2 = \delta m^2$ , which follow from the identity  $\Gamma^{(2)}(x, y) = iG^{-1}(x, y; \phi = 0)$ , as well as  $\delta \lambda_2 = \delta \lambda_1$ , which follows from  $V(x, y, z, t) = \bar{V}(x, y, z, t)$ .<sup>13</sup> Notice that, although these identities are true in the exact theory, they may not be satisfied in general at the level of approximations.

before, non-secular approximation respecting late-time universality can be obtained from systematic expansion schemes of higher effective actions. In general this leads to non-linear integro-differential equations for the  $n$ -point functions, which are now independent self-adjusting variables. As described previously, this is a crucial ingredient for late-time universality.

The use of  $n$ PI effective actions with  $n > 2$  can be important to describe non-Gaussian initial density matrix for non-equilibrium systems. They are also relevant in the context of high-temperature gauge theories for a quantitative description of transport coefficients beyond the “leading-log” approximation,<sup>27</sup> or in the context of critical phenomena near second-order phase transitions. For instance, the quantitative description of the critical exponents of scalar  $\phi^4$  theory goes beyond any finite order 2PI loop-expansion<sup>h,14</sup> It requires taking into account vertex corrections that start with the 4PI effective action to four-loop order.

For practical purposes, a crucial observation is that there exists an equivalence hierarchy between  $n$ PI effective actions. For instance, in the case of a loop expansion one has:<sup>27</sup>

$$\begin{aligned}\Gamma^{(1\text{loop})}[\phi] &\neq \Gamma^{(1\text{loop})}[\phi, G] = \Gamma^{(1\text{loop})}[\phi, G, V_3] = \dots, \\ \Gamma^{(2\text{loop})}[\phi] &\neq \Gamma^{(2\text{loop})}[\phi, G] = \Gamma^{(2\text{loop})}[\phi, G, V_3] = \Gamma^{(2\text{loop})}[\phi, G, V_3, V_4] = \dots \\ \Gamma^{(3\text{loop})}[\phi] &\neq \Gamma^{(3\text{loop})}[\phi, G] \neq \Gamma^{(3\text{loop})}[\phi, G, V_3] = \Gamma^{(3\text{loop})}[\phi, G, V_3, V_4] = \dots \\ &\vdots\end{aligned}$$

where  $\Gamma^{(n\text{loop})}$  denotes the approximation of the respective effective action to  $n$ -loop order in the absence of external sources. E.g. for a two-loop approximation all  $n$ PI descriptions with  $n \geq 2$  are equivalent and the 2PI effective action captures already the complete answer for the self-consistent description up to this order. In contrast, a self-consistently complete result to three-loop order requires at least the 3PI effective action in general, etc.<sup>i</sup> Typically the 2PI, 3PI or maybe the 4PI effective action captures already the complete answer for the self-consistent description to the desired/computationally feasible order of approximation.

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<sup>h</sup>Critical phenomena can be described using the 2PI  $1/N$ -expansion beyond LO.<sup>14</sup>

<sup>i</sup>There exist further simplifications which can decrease the hierarchy even further such that lower  $n$ PI effective actions can be sufficient in practice. For instance, for a vanishing field  $\phi$ , one has, in the absence of sources:  $\Gamma^{(3\text{loop})}[\phi = 0, G] = \Gamma^{(3\text{loop})}[\phi = 0, G, V_3 = 0, V_4] = \dots$

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